Directions: Complete all question and **show all applicable work.** Partial credit will be given.

1.) The angle of elevation of the sun is decreasing at a rate of 0.25 radians per hour. How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun is $\pi/6$?

2.) Find any vertical and horizontal asymptotes, maximum and minimum values, intervals of concavity and inflection point(s) of $f(x) = x^4 + 4x^3$ on the interval $[-5, 5]$. Create a graph of $f(x)$ labeling the properties found above.

3.) A metal storage tank is to be constructed with volume V in the shape of a right circular cylinder with half spheres closing each end. What dimensions in terms of V will use the least amount of metal?

4.) Calculate:

a.) $\lim_{x\to\infty} \frac{x^2}{e^{2x}}$

b.) $\lim_{x\to\infty} x - x\ln(x)$

c.) $\lim_{x\to\frac{\pi}{2}^-} (\tan x)^{cosx}$

5.) A jet traveling at 0.2 km/s flies directly over a ground-based radar station at horizontal height 3 km. How fast is the distance between the plane and the station increasing 20 seconds later?

6.) A ball is thrown vertically upward at 20 m/s, from a height of 5 meters. Assuming the the acceleration due to gravity is 9.8 m/s^2,

- Find a function describing the velocity at any time t.
- Find a function describing the displacement at any time t.
- When does the ball reach its peak height? How high is this?
- How long until the ball hits the ground?

Answers

1.)
2.)
3.) From class we know the relevant equations are

$$V = \pi r^2 h + \frac{4}{3}\pi r^3 \quad \text{and} \quad A = 4\pi r^2 + 2\pi r h$$

Since the goal is to minimize survace area (A), the right hand side needs to be simplified such that r or h is subsituted by an equaivlent function of the other variable. Since h can easily be solved for in the volume equation:

$$V = \pi r^2 h + \tfrac{4}{3}\pi r^3$$
$$h = \frac{V - \frac{4}{3}\pi r^3}{\pi r^2}$$

and substituting into the surface area equation gives:

$$A = 4\pi r^2 + 2\pi r \left(\frac{V - \frac{4}{3}\pi r^3}{\pi r^2} \right)$$

which simplifies to:

$$A = \frac{4}{3}\pi r^2 + \frac{2V}{r}.$$

Since the surface area is to be minimized the following three cases need to be dealt with:

- Places where the deriviative does not exist.
- Critical points (where the deriviative = 0)
- end points.

There are no relevant endpoints to consider. While the radius needs to be positive, and if near zero, the height would tend towards infinity (and obviously not be a ideal solution). Likewise it the radius were to tend towards infinity, the surface area would tend towards inifinty. Thus we can ignore endpoints. Along the same lines, there is one point were the deriviative does not exist, and that is when $r = 0$, but this may be ignored for similar reasoning.

This simplifies the problem to just critical points where the deriviative equals zero.

$A = \frac{4}{3}\pi r^2 + \frac{2V}{r}$	take deriviative
$\frac{dA}{dr} = \frac{4}{3}\pi(2)r + (-1)\frac{2V}{r^2} = 0$	simplify
$\frac{8}{3}\pi r - \frac{2V}{r^2} = 0$	solve for r
$\frac{\frac{8}{3}\pi r^3 - 2V}{r^2} = 0$	solve for r (cont.)
$\frac{8}{3}\pi r^3 - 2V = 0$	solve for r (cont.)
$r = \sqrt[3]{\frac{2V}{\frac{8}{3}\pi}}$	solve for r (cont.)
$r = \sqrt[3]{\frac{3V}{4\pi}}$	solve for r (cont.)

This gives a critical point. Then next step is to determine if it is a local max or min. This is done by the second deriviative.

$\frac{dA}{dr} = \frac{8}{3}\pi r - \frac{2V}{r^2}$	Take 2nd deriviative
$\frac{d^2A}{dr^2} = \frac{8}{3}\pi + \frac{4V}{r^3}$	

Now plug $r = \sqrt[3]{\frac{3V}{4\pi}}$ into the above equation.

$\frac{d^2A}{dr^2} = \frac{8}{3}\pi + \frac{4V}{\left(\sqrt[3]{\frac{3V}{4\pi}}\right)^3}$	Now simplify
$\frac{d^2A}{dr^2} = \frac{8}{3}\pi + \frac{16\pi}{3} > 0$	

The second deriviative is positive, so the function is always concave up at the critical point. Using the 2nd deriviative test, the point $r = \sqrt[3]{\frac{3V}{4\pi}}$ is a local minimum. Thus the optimal storage tank has dimensions:

$$r = \sqrt[3]{\frac{3V}{4\pi}} \qquad \text{and}$$
$$h = \frac{V - \frac{4}{3}\pi r^3}{\pi r^2} \qquad \text{now sub in } r$$
$$h = \frac{V - \frac{4}{3}\pi\left(\sqrt[3]{\frac{3V}{4\pi}}\right)^3}{\pi\left(\sqrt[3]{\frac{3V}{4\pi}}\right)^2} \qquad \text{and finally to simplify}$$
$$h = \frac{V - V}{\pi\left(\sqrt[3]{\frac{3V}{4\pi}}\right)^2} \qquad \text{and finally to simplify}$$
$$h = 0$$

Thus the optimal shape is not what may have been expected. The optimal tank has $h = 0$, which means there is no cylindrical part, and only two half-spheres, connected directly (ie. a sphere!).

4.)

5.) Begin by drawing a right triangle, connecting the ground station, the imaginary point 3 km above station and the current position of the plane.

Step 2: Find how far the plane has travelled horizontally. If the plane is flying at 0.2 km/s then 20 seconds later the plane will have travelled $0.2 * 20 = 4km$. Add this to the triangle.

Step 3: Find the current distance from the station to the plane. Then using pythagorean theorem the third side is 5 km.

Step 4: Now that the necessary distances are found, we are looking for the change in distance of the plane to the ground station. The horizontal change (speed) is given, so begin with the pythagorean theorem and use implicit differentation.

$$a^2 + b^2 = c^2 \qquad \text{take deriviative w.r.t. time (t).}$$
$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

The following variables are taken from the problem:

- The height is 3 km (a = 3)
- There is no change in the height ($\frac{da}{dt} = 0$)
- The horizontal distance is 4 km (b = 4)
- The horizontal flying speed is 0.2 km/s ($\frac{db}{dt} = 0.2$)
- The distance from the ground station and plane is 5 km (c = 5)
- The change in distance from the ground station to the plane is the desired unit ($\frac{dc}{dt}$)

Solving gives:

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$
$$2(3)(0) + 2(4)(0.2) = 2(5)\frac{dc}{dt}$$
$$10\frac{dc}{dt} = \frac{8}{5}$$
$$\frac{dc}{dt} = \frac{8}{50}$$
$$\frac{dc}{dt} = 0.16km/s$$

Thus the plane's distance from the ground station is increasing at a rate of 0.16 km/s.

6.) $a(t) = -9.8\frac{m}{s^2}$, so $v(t)$ can be found using the antiderivative.[1] The antiderivative is a function whose deriviative is -9.8. This is a reverse power rule equation. Let $n = 0$, so

$$v(t) = -9.8t + C = -9.8t + C.$$

Note that we use the variable t for time here.

To solve for C, we note the initial velocity is $20m/s$ upward (positive). This happens when $t = 0$. So:

$$\begin{aligned} v(t) &= -9.8t + C && \text{Now sub in } t = 0 \text{ and } v(0) = 20 \\ 20 &= -9.8(0) + C && \text{Simplify} \\ C &= 20 \end{aligned}$$

So the equation becomes:

$$v(t) = -9.8t + 20.$$

Part B: Now we move onto the displacement $s(t)$. This is the antiderivative of $v(t)$, again using the reverse power rule. For the first term, $n = 1$ and the second $n = 0$, so:

$$\begin{aligned} s(t) &= -9.8\left(\tfrac{1}{2}\right)t^2 + 20t + C_2 \\ s(t) &= -4.8t^2 + 20t + C_2 && \text{Now sub in } t = 0 \text{ and } s(0) = 5 \text{ meters} \\ 5 &= -4.9(0)^2 + 20(0) + C_2 && \text{Simplify} \\ C_2 &= 5 \end{aligned}$$

so the displacement equation becomes:

$$s(t) = -4.9t^2 + 20t + 5.$$

Part C: When does the ball reach the peak height? The ball reaches the peak height when the ball's veloctity changes from moving upwards (positive) to downwards (negative). Namely $v(t) = 0$. So set $v(t) = 0$ and solve for t:

$$\begin{aligned} v(t) &= 0 \\ 0 &= -9.8t + 20 \\ t &= 2.04 \text{ seconds} && \text{(rounded)} \end{aligned}$$

This gives the time (which is very reasonable). The question also asks for the height, so compute:

$$s(2.04) = -4.8(2.04)^2 + 20(2.04) + 5 = 25.82 \text{ meters}$$

We move on to the final part.

Part D: How long does it take for the ball to hit the ground? The ball hits the ground when it's displacement is zero. So $s(t) = 0$.

$$\begin{aligned} s(t) &= 0 \\ 0 &= -4.8t^2 + 20t + 5 && \text{Now use quadractic formula} \\ t &= \tfrac{-20 \pm \sqrt{(20)^2 - 4(-4.8)(5)}}{2(-4.8)} \\ t &= \tfrac{-20 \pm \sqrt{496}}{2(-4.8)} \\ t &= 4.40 \text{ and } t = -.24 \end{aligned}$$

The negative time is irrelevant to this problem. The positive t gives the time, namely 4.4 seconds after being thrown, again very reasonable.

[1] It is -9.8 because gravity points down.