Questions to go over:

- 3.1 # 13
- 3.2 # 26
- 3.3 # 21
- 3.4 # 14, 28
- 3.5 # 16 - See footnote.[1]
- 3.6 # 4 - Solve using undetermined coefficients, the long variation of parameters method and formula for variation of parameters. See footnote.[2]
- 3.7 # 11 - Ignore the 2nd part discussing frequency.
- 3.8 # 10 - Attempt, but do not dwell on the 2nd part.

This test covers chapter 3, which is Second Order Linear Equations in the form of:

$$y'' + p(t)y' + q(t)y = g(t)$$

with initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$. Most of the chapter dealt with constant coefficients and the homogeneous case, namely:

$$ay'' + by' + cy = 0$$

with initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$.

Non-homogeneous, non-constant coefficients, and applications are also discussed.

1 Constant coefficient, homogeneous equations

Given

$$ay'' + by' + cy = 0$$

with initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$, the solution can be found by rewriting this as a second order polynomial equation:

$$ar^2 + br + c = 0$$

where the solution is given by the quadratic formula:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the roots are **real and different** a general solution is then:

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

[1] This is a bit tricky because the RHS consists of two functions *multiplied together*. If they were added, you could treat each separately like on the quiz, but because they are multiplied we need the general form of a first order polynomial times an exponential: $y = (At + B)e^{2t}$. Note that a "C" is unnecessary on the exponential because you can factor it out front and combine it with the other constants A and B.

[2] Expect to solve a problem with Method of Undetermined Coefficients, then solve the same problem with Variation of Parameters. The answers should be the same, but you may have to use some identities/constant manipulation to make them look identical. eg. $c_1 e^{4t} + 3 + 2e^{4t} = c_1 e^{4t} + 3$, if you factor out a e^{4t}, you can combine the constants into a new constant. They are both correct though.

if the roots are **real and the same** a general solution is:

$$y = c_1 e^{rt} + c_2 t e^{rt},$$

if the roots are **complex**, which means they are in the form $r_{1,2} = \lambda \pm i\mu$, a general solution is:

$$y = c_1 e^{r_1 t}\cos(\mu t) + c_2 e^{r_2 t}\sin(\mu t).$$

Each of these three has a distinctive shape. Particularly the last oscillates, with either increasing, steady or decreasing (depending on the exponential) magnitude. The first two typically tend towards zero or infinity exponentially depending on the exponents of the exponential.

2 Solution existence, Uniqueness, Fundamental Solutions, The Wronskian, and the General Solution

If a problem can be written in the form:

$$y'' + p(t)y' + q(t)y = g(t)$$

with initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$, and p, q and g are continuous on an open interval I containing the initial condition, then by Thm 3.2.1 there is exactly one solution throughout I.

This means that if we find a solution (and it does exist), it is unique, assuming it satisfied the assumption of continuity. Further by Thm 3.2.4 if we find a tentative solution of the form $y = c_1y_1 + c_2y_2$, we can show this solution is the general solution to the DE if the Wronskian $W = y_1y_2' - y_1'y_2 \neq 0$. If the Wronskian is zero, there must be another solution (as it exists), but does not give any insight on how to find it. Finding such a solution is the task of chapter 3.4 (reduction of order), 3.6 (undetermined coefficients) and 3.7 (variation of parameters).

3 Reduction of order

Similar to how higher order polynomials can be reduced in order if a factor is known (by polynomial division), 2nd order DE can be reduced in order if a solution is known. Reduction of order takes a known solution y_1 and assumes a new solution in the form:

$$y_2 = v(t)y_1$$

where $v(t)$ is a general function.

y_2, y_2', and y_2'' are each calculated and plugged into the DE. After some simplification a new simpler DE if formed involving $v(t)$ which then can be solved.

This solution must be checked that it adds new information, namely that the Wronskian is non-zero. If the Wronskian is non-zero, then the problem is solved and the general solution is:

$$y = c_1y_1 + c_2v(t)y_1(t).$$

Note that this equation can typically be simplified similar to footnote 2 found on page 1.

4 Non-homogeneous equations: Method of undetermined coefficients

Given a second order non-homogeneous equation, there are three things to be done.

1. Solve the homogeneous equation,

2. Guess the general form of the right-hand side and find the particular solution, and
3. Combine these to get the general solution: $y = c_1 y_1 + c_2 y_2 + y_3$, where the first two terms are the general solution to the homogeneous equation and the third term is the particular solution for the RHS.

Part 2 is the only new task. The guess should be a *general* form of the right hand side. For example:

- If RHS is $4\sin(3x)$, then guess general $\sin + \cos$ with same argument: $A\sin(3x) + B\cos(3x)$.
- If RHS is $3t^2 + t$, then guess general 2nd order polynomial: $At^2 + Bt + C$.
- If RHS is $5e^3 t$, then guess general exponential with same exponent: Ae^{3t}.

If there are two functions being added, they can be treated separately. If two functions are multiplied, they need to be treated together (See footnote 1 on 1st page).

5 Non-homogeneous equations: Variation of parameters

Solving non-homogeneous equations where the RHS cannot be easily guessed can be accomplished through variation of parameters. Here, similar to reduction of order above, the c_1 and c_2 in the homogeneous solutions are assumed to be be function of t. The first derivative and second derivative of this "solution" is taken and plugged into the DE.

After much algebra and assuming $u_1' y_1' + u_2' y_2' = 0$ the DE reduces to a linear equation in u_1' and u_2'. This assumption and linear equation provide two equations to solve for the unknowns u_1' and u_2'. Integrate to find u_1 and u_2 (do not forget to get a c_1 and c_2 from the integration). You can verify the DE is satisfied by plugging this new solution into the DE. If the original solution y_1 and y_2 formed a fundamental set (Wronskian $\neq 0$), then this new solution will also be the general solution provided the c_1 and c_2 were created from the integration.

There is a closed form solution to this problem as described in thm 3.6.1 (pg. 188). Please understand how to do simple problems the long way and harder problems using the thm. I would suggest re-working the problem from the notes and doing the example problem from the text 185-187, using both the long way and the formula.

6 Springs - with/without dampening and external forces

Section 3.7 and 3.8 discuss springs (among other things). I would like you to be able to setup a spring problem and be able to explain what a solution is telling you. See your notes for some problems (be sure you can set these up reasonably quickly). Solving them comes from the earlier parts of the chapter. Finally you must analyze the solutions. For example, a solution:

- $u(t) = 3e^{3t} - 3e^{-2t}$ is a spring that will drop forever ($3e^{3t}$ is a positive (which is down) exponential, the other term dies out).
- $u(t) = 3e^{3t}\cos(4t) + 2e^{3t}\sin(4t)$ is a spring that will oscillate ($\sin/\cos$) growing in magnitude (e^{3t}). This obviously cannot happen forever in reality, but this is what the model says.
- $u(t) = 0$ is a spring at equilibrium.

Setup: Remember the key ideas:

- First that the spring exerts a force in the opposite direction as gravity and at rest $mg = kL$. This can be used to solve for the spring constant.

- If mass is not given but weight is: $F = ma$ so weight $= mg$.
- The dampening force is dependent on velocity, so dampening force $= -\gamma u'$.
- The units need to align, and selecting feet for English and meters for metric is advised due to how gravity is defined.
- The parameters must be all be positive. m and k should always be positive from the above equations, γ should be positive from the equations as well, but if the question is read incorrectly a common occurrence is to end up with the negative of the correct positive answer.

You are **not** required to know about the frequency, period, etc. from these chapters, or the conversion to R. We will come back to this at the end of the semester if we have time.